A METHOD OF SOLVING BOUNDARY VALUE PROBLEMS IN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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A solution in the form of a series in time functions and polynomials of the space coordinate is obtained using as an example the moisture conduction equation with allowance for the finite rate of capillary motion.

The equation of moisture conduction in capillary-porous bodies for an infinite plate with account for the finite rate of capillary motion has the form

$$\frac{\partial u}{\partial \mathrm{Fo}} + \mathrm{Fo}^* \frac{\partial^2 u}{\partial \mathrm{Fo}^2} = \frac{\partial^2 u}{\partial x^2}. \quad (1)$$

The solution of Eq. (1) is found in series form

$$u(x, \mathrm{Fo}) = \sum_{n=0}^{\infty} P_n(x)\varphi^{(n)}(\mathrm{Fo}) + \sum_{k=0}^{\infty} Q_k(x)\psi^{(k)}(\mathrm{Fo}), \quad (2)$$

where $\varphi(\mathrm{Fo})$, $\psi(\mathrm{Fo})$ are arbitrary functions of time, $P_n(x)$, $Q_k(x)$ are polynomials whose form is determined from the boundary conditions.

We will consider the solution of Eq. (1) for an asymmetric plate with the condition

$$u(0, \mathrm{Fo}) = \Omega(\mathrm{Fo}) \quad (3)$$

at one surface, one of the conditions

$$u(1, \mathrm{Fo}) = \omega(\mathrm{Fo}), \quad (4)$$

$$\frac{\partial u(1, \mathrm{Fo})}{\partial x} = \chi(\mathrm{Fo}), \quad (5)$$

$$-\frac{\partial u(1, \mathrm{Fo})}{\partial x} + \mathrm{Bi}\left[u_c(\mathrm{Fo}) - u(1, \mathrm{Fo})\right] = 0 \quad (6)$$

at the other surface, and the initial conditions

$$u(x, 0) = \Phi(x), \; \frac{\partial u(x, 0)}{\partial \mathrm{Fo}} = F(x). \quad (7)$$

Substitution of (2) in (1) gives the relations

$$P''_0(x) = 0, \; P_0 = P''_1(x), \; P_{n-1}(x) + \mathrm{Fo}^* P_{n-2}(x) = P''_n(x), \quad (8)$$

$$Q''_0(x) = 0, \; Q_0 = Q''_1(x), \; Q_{k-1}(x) + \mathrm{Fo}^* Q_{k-2}(x) = Q''_k(x)$$

$$n = 2, 3, \ldots, \quad k = 2, 3, \ldots \quad (9)$$

1. We obtain the solution of (1) for conditions (3), (4). We set

$$\varphi(\mathrm{Fo}) = \Omega(\mathrm{Fo}), \; \psi(\mathrm{Fo}) = \omega(\mathrm{Fo}). \quad (10)$$

Substituting (2) in (3) and (4), and using (10), we obtain

$$P_0(0) = 1, \; P_{n-1}(0) = 0, \; Q_{k-2}(0) = 0; \quad (11)$$

$$Q_0(1) = 1, \; Q_{k-1}(1) = 0, \; P_{n-2}(1) = 0. \quad (12)$$

Integrating (8) and (9) and using (11) and (12), we have

$$P_n(x) = \int_0^x dx_1 \int_0^{x_1} \left[P_{n-1}(\xi) + \mathrm{Fo}^* P_{n-2}(\xi)\right] d\xi - x\int_0^1 dx_1 \int_0^{x_1} \left[P_{n-1}(\xi) + \mathrm{Fo}^* P_{n-2}(\xi)\right] d\xi; \quad (13)$$

$$Q_k(x) = \int_0^x dx_1 \int_0^{x_1} \left[Q_{k-1}(\xi) + \mathrm{Fo}^* Q_{k-2}(\xi)\right] d\xi - x\int_0^1 dx_1 \int_0^{x_1} \left[Q_{k-1}(\xi) + \mathrm{Fo}^* Q_{k-2}(\xi)\right] d\xi; \quad (14)$$

$$P_0(x) = -x + 1, \; Q_0(x) = x; \quad (15)$$

$$P_1(x) = -x^3/6 + x^2/2 - x/3, \; Q_1(x) = x^3/6 - x/6. \quad (16)$$

2. We consider conditions (3) and (5). We set

$$\varphi(\mathrm{Fo}) = \Omega(\mathrm{Fo}), \; \psi(\mathrm{Fo}) = \chi(\mathrm{Fo}). \quad (17)$$

Substituting (2) in (3) and (5) and using (17), we obtain

$$P_0(0) = 1, \; P_{n-1}(0) = 0, \; Q_{k-2}(0) = 0, \quad (18)$$

$$P'_{n-2}(1) = 0, \; Q'_0(1) = 1, \; Q'_{k-1}(1) = 0. \quad (19)$$

Integrating (8) and (9) and using (18) and (19), we find

$$P_n(x) = \int_0^x dx_1 \int_0^{x_1} \left[P_{n-1}(\xi) + \mathrm{Fo}^* P_{n-2}(\xi)\right] d\xi - x\int_0^1 \left[P_{n-1}(\xi) + \mathrm{Fo}^* P_{n-2}(\xi)\right] d\xi; \quad (20)$$

$$Q_k(x) = \int_0^x dx_1 \int_0^{x_1} \left[Q_{k-1}(\xi) + \mathrm{Fo}^* Q_{k-2}(\xi)\right] d\xi - x\int_0^1 \left[Q_{k-1}(\xi) + \mathrm{Fo}^* Q_{k-2}(\xi)\right] d\xi; \quad (21)$$

$$P_0(x) = 1, \; Q_0(x) = x; \quad (22)$$

$$P_1(x) = -x^3/6 + x^2/2 - x/2, \; Q_1(x) = x^3/6 - x/2. \quad (23)$$

3. We consider conditions (3) and (6). We set

$$\varphi(\mathrm{Fo}) = \Omega(\mathrm{Fo}), \; \psi(\mathrm{Fo}) = u_c(\mathrm{Fo}). \quad (24)$$

Substituting (2) into (3) and (6) and using (24), we obtain

$$P_0(0) = 1, \; P_{n-1}(0) = 0, \; Q_{k-2}(0) = 0; \quad (25)$$

$$P'_{n-2}(1) + \mathrm{Bi}P_{n-2}(1) = 0, \; Q'_0(1) + \mathrm{Bi}\,Q_0(1) = \mathrm{Bi}; \quad (26)$$

$$Q'_{k-1}(1) + \mathrm{Bi}\,Q_{k-1}(1) = 0. \quad (27)$$

Integrating (8) and (9) and using (25)–(27), we obtain

$$P_n(x) = \int_0^x dx_1 \int_0^{x_1} \left[P_{n-1}(\xi) + \mathrm{Fo}^* P_{n-2}(\xi)\right] d\xi - \frac{x}{\mathrm{Bi}+1} \times$$

$$\times\left\{\mathrm{Bi}\int_0^1 dx_1\int_0^{x_1}[P_{n-1}(\xi)+\mathrm{Fo}^*P_{n-2}(\xi)]\,d\xi+\right.$$
$$\left.+\int_0^1[P_{n-1}(\xi)+\mathrm{Fo}^*P_{n-2}(\xi)]\,d\xi\right\}; \tag{28}$$

$$Q_k(x)=\int_0^x dx_1\int_0^{x_1}[Q_{k-1}(\xi)+\mathrm{Fo}^*Q_{k-2}(\xi)]\,d\xi-\frac{x}{\mathrm{Bi}+1}\times$$
$$\times\left\{\mathrm{Bi}\int_0^1 dx_1\int_0^{x_1}[Q_{k-1}(\xi)+\mathrm{Fo}^*Q_{k-2}(\xi)]\,d\xi+\right.$$
$$\left.+\int_0^1[Q_{k-1}(\xi)+\mathrm{Fo}^*Q_{k-2}(\xi)]\,d\xi\right\}; \tag{29}$$

$$P_0(x)=\frac{-\mathrm{Bi}\,x}{1+\mathrm{Bi}}+1,\ P_1(x)=\frac{-\mathrm{Bi}\,x^3}{(1+\mathrm{Bi})\cdot 6}+$$
$$+\frac{x^2}{2}-\frac{x(\mathrm{Bi}^2+3\mathrm{Bi}+3)}{3(\mathrm{Bi}+1)^2}; \tag{30}$$

$$Q_0(x)=\frac{\mathrm{Bi}x}{\mathrm{Bi}+1},$$
$$Q_1(x)=\frac{\mathrm{Bi}}{6(\mathrm{Bi}+1)}\left[x^3-\frac{x(3+\mathrm{Bi})}{\mathrm{Bi}+1}\right]. \tag{31}$$

4. Solutions of (1) obtained in the form of series (2) with conditions (3) and any of conditions (4)–(6) do not satisfy the initial conditions (7).

Indeed,

$$u(x,0)=\sum_{n=0}^{\infty}P_n(x)\varphi^{(n)}(0)+$$
$$+\sum_{k=0}^{\infty}Q_k(x)\psi^{(k)}(0)=\overline{\Psi}_1(x); \tag{32}$$

$$\frac{\partial u(x,0)}{\partial \mathrm{Fo}}=\sum_{n=0}^{\infty}P_n(x)\varphi^{(n+1)}(0)+$$
$$+\sum_{k=0}^{\infty}Q_k(x)\psi^{(k+1)}(0)=\overline{\Psi}_2(x). \tag{33}$$

We can satisfy the initial conditions if we add to the solution (3) the solution $\bar{u}(x,\mathrm{Fo})$ satisfying homogeneous boundary conditions and the initial conditions

$$\bar{u}(x,0)=\Psi_1(x),\ \frac{\partial\bar{u}(x,0)}{\partial \mathrm{Fo}}=\Psi_2(x), \tag{34}$$

where

$$\Psi_1(x)=\Phi(x)-\overline{\Psi}_1(x),\ \Psi_2(x)=F(x)-\overline{\Psi}_2(x). \tag{35}$$

Finally, the solution of (1) satisfying conditions (3) and (7) and any of the conditions (4)–(6) takes the form

$$u(x,\mathrm{Fo})=\sum_{n=0}^{\infty}P_n(x)\varphi^{(n)}(\mathrm{Fo})+$$
$$+\sum_{k=0}^{\infty}Q_k(x)\psi^{(k)}(\mathrm{Fo})+\bar{u}(x,\mathrm{Fo}). \tag{36}$$

The solution of (1) in form (36) is convenient for approximate calculations, for analyzing the effect of variable boundary conditions (which is difficult in the case of solutions obtained in some other form), and for determining thermophysical characteristics [1]. In conclusion we note that the arbitrary functions $\varphi(\mathrm{Fo})$ and $\psi(\mathrm{Fo})$ can be given different physical meanings. In this paper they are set equal to functions given on the boundary. In other cases one of them may be treated as the average integral moisture content or as a measure of the moisture content at some interior point, etc. Expansion (36) can also be used to solve inverse problems [2].

REFERENCES

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